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ON THE THEORY OF REPRODUCING KERNEL HILBERT SPACES

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Abstract. The inner product in RKHS is described in abstract form. Some of the results, published earlier, are discussed from a general point of view. In particular, the characterization of the range of linear integral transforms and inversion formulas, announced in the works of Saitoh, are analyzed.

1. Introduction.

The theory of reproducing kernels was developed in [A], [S]. A recent review of the theory is [Sa1,Sa2], where the reader can find many references.

The basic result in [A] is the existence and uniqueness of a reproducing kernel Hilbert space (RKHS) corresponding to any *self-adjoint nonnegative-definite kernel* $K(p,q)$, $p,q \in E$, where E is an abstract set. Let H be a Hilbert space of functions defined on E, and $H \subset L^2(E)$. Assume that $K(\cdot,q)$ and $K(p,\cdot)$ belong to H. Let us assume that the linear operator $K : H \to H$, with the kernel $K(p,q)$, is injective. It is defined on all of H since $K(p,\cdot) \in H$ by the assumption. Define RKHS H_K inner product by the formula

$$(f,g)_{H_K} := [f,g] := (K^{-1}f,g), \tag{1.1}$$

where $(f,g) := (f,g)_{L^2(E)}$, K^{-1} is the operator inverse to $K : H \to H$, and

$$Kf := \int_E K(p,q)f(q)dq. \tag{1.2}$$

The injectivity assumption can be dropped, but then one has to consider K on the factor space $H/N(K)$, where $N(K) := \{f : Kf = 0\}$ is the null-space of K.

In the literature (e.g. see [Sa1,2]) the inner product in RKHS was not defined explicitly by formula (1.1). The definition of the inner product in H_K, given in [A] (and presented in [Sa1, p.36]) is implicit and contains some limiting procedure which is not described explicitly. In particular, it is not clear over which sets of p and q the summation in formula (11) in [Sa1, p.36] is taken. In [A] such a summation is taken over a finite set of points $p \in E$ and $q \in E$. The finite sums $\sum_p X_p K(\cdot,p)$, used in [Sa1, p.36] do not form a complete Hilbert space H_K, and the completion procedure is not discussed in sufficient details in [Sa1]. Our definition (1.1) of the inner product in H_K coincides with the definition in [Sa1, p.36, formula (11)] if one takes f and g in (1.1) to be finite linear combinations of the functions of the type $K(p,\cdot)$ and $K(\cdot,q)$.

The reproducing property of the kernel $K(p,q)$ can be stated as follows:

$$[f(\cdot), K(\cdot,q)] = f(q), \tag{1.3}$$

and this formula can be easily derived from the definition (1.1) of the inner product in H_K:

$$[f(\cdot), K(\cdot,q)] := (K^{-1}f, K(\cdot,q)) = (f, K^{-1}K(\cdot,q)) = (f, I(\cdot,q)) = f(q). \tag{1.4}$$

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Here we have used the selfadjointness of the operator K^{-1}, and the fact that the distributional kernel of the identity operator I is $\delta(p-q)$, the delta function, which is well defined on RKHS because the value $f(p)$ for any $p \in E$ is a bounded linear functional in H:

$$|f(q)| \leq ||f||\, ||K(\cdot, q)||, \tag{1.5}$$

where $||f|| := [f, f]^{\frac{1}{2}}$ is the norm in H_K.

The basic results of this paper are:

1) representation of the inner product in H_K by formula (1.1),

and

2) clarification of the conditions from [Sa1,2] under which the range of the general linear transform, defined by formula (2.1) below, is characterized and inversion formulas for this transform are obtained.

2. Linear transforms and RKHS.

Define

$$f(p) := LF := \int_T \overline{h(t,p)} F(t) dm(t), \tag{2.1}$$

where $T \subset \mathbb{R}^n$ is some subset of $\mathbb{R}^n$, $dm(t)$ is a positive measure on T, and $h(t,p)$ is a function on $H_0 \times H$, where $H_0 := L^2(T, dm(t))$. The linear operator $L : H_0 \to H$ is injective if the set $\{h(t,p)\}_{\forall p \in E}$ is total in H_0. This means that if for some $F \in L^2(T, dm(t))$ the following equation holds:

$$0 = \int_T h(t,p)\, F(t) dm(t) \qquad \forall p \in E, \tag{2.2}$$

then $F(t) = 0$.

Let us assume that L is injective. The operator $L^* : H \to H_0$ acts by the formula:

$$(LF, g)_H = (F, L^* g)_{H_0},$$

thus

$$L^* g = \int_E h(t,p) g(p) dp. \tag{2.3}$$

Recall that we assume in this paper that K and L are injective, so that K^{-1} and L^{-1} exist. Let us state a simple lemma.

Lemma 2.1. *One has*

$$[LF, LG] = (F, G)_{H_0}, \tag{2.4}$$

provided that RKHS H_K is defined by the kernel

$$K(p,q) := \int_T \overline{h(t,p)} h(t,q) dm(t). \tag{2.5}$$

Proof. One has

$$[LF, LG] = (K^{-1}LF, LG) = (L^* K^{-1} LF, G)_{H_0} = (F, G)_{H_0}, \tag{2.6}$$

where the operator L in (2.6), after the first equality sign, is considered as an operator from H_0 into H. The last step in (2.6) is based on the relation:

$$L^* K^{-1} L = I. \tag{2.7}$$

Let us assume that L^{-1} is a closed, possibly unbounded, densely defined operator from $R(L) \subset H$ into H_0, where $R(L)$ is the range of L. Then formula (2.7) is equivalent to the relation:

$$K = LL^*. \tag{2.8}$$

Indeed, in this case one has:

$$K^{-1} = (LL^*)^{-1} = L^{*-1}L^{-1}, \tag{2.9}$$

so that (2.7) and (2.8) are equivalent.

Note that under our assumptions about L^{-1} the operator $(L^*)^{-1}$ does exist and $(L^*)^{-1} = (L^{-1})^*$.

Let us prove that (2.5) is equivalent to (2.8) and, consequently, to (2.7). Using (2.1) and (2.3), one gets:

$$LL^*g = \int_T \overline{h(t,p)} \int_E h(t,q)g(q)dq\, dm(t) = \int_E K(p,q)g(p)dp, \tag{2.10}$$

where $K(p,q)$ is defined by (2.5). Since $g(p)$ in (2.10) is arbitrary, this formula implies (2.8), as claimed. Therefore (2.5) implies (2.8), and, consequently, (2.7), and (2.7) implies (2.4) according to (2.6). Lemma 2.1 is proved. $\square$

In [Sa1] it is proposed to characterize the range $R(L)$ of linear map (2.1) as the RKHS with the reproducing kernel (2.5). It follows from Lemma 2.1 that if one puts the inner product (1.1) of H_K, with $K(p,q)$ defined in (2.5), on the set $R(L)$, then $L : H_0 \to H_K$ is an isometry (see (2.6)).

In general one cannot describe the norm in H_K in terms of some standard norms, such as the Sobolev norm.

Therefore the above observation (that $R(L) = H_K$ if one puts the norm of H_K onto $R(L)$) does not solve the problem of characterization of the range of $L : H_0 \to H_0$ as an operator from H_0 into H_0.

This point was discussed in [R2]. On the other hand, some cases are known when one can characterize the norm in H_K in terms of the Sobolev norms (positive or negative) [R1].

It is also claimed in [Sa1,2] that an inversion formula exists for a general linear transform (2.1) ([Sa2, p.56, formula (31)]).

This inversion formula is derived under the assumption [Sa2, p.58] that H_K is the space $L^2(E, d\mu)$, where $d\mu$ is some positive measure. This assumption means that the kernel $A(p,q)$ of the operator K^{-1} is a distribution of the form $\delta(p-q)w(p)$, where $w(p)$ is the density of the measure $d\mu(p)$, that is $d\mu(p) = w(p)dp$, and $\delta(p-q)$ is the delta function.

This and the definition of the inverse operator, namely $KK^{-1} = I$, written in terms of kernels, imply:

$$\delta(p-q) = \int_E \delta(p-s)K(s,q)d\mu(s) = K(p,q)w(p), \tag{2.11}$$

where we have assumed that $w(p) > 0$ is a smooth function, with $v(p) := \frac{1}{w(p)} > 0$. Thus (2.11) implies that the reproducing kernel $K(p,q)$ must be of the form:

$$K(p,q) = v(p)\delta(p-q), \tag{2.12}$$

if one assumes that the inner product in H_K is the same as in $L^2(E, d\mu)$, as indeed S.Saitoh assumes in [Sa2, p.56] and in [Sa1].

Assumption (2.12) is not satisfied in general, and is essentially equivalent to the formula $L^{-1} = L^*$, where L now is an operator from H_0 into H_K.

Let us prove the above claim. If L is considered as operator from H_0 into H_K, then formula (2.7) can be written as

$$L^*L = I, \quad L : H_0 \to H_K \tag{2.13}$$

and formula (2.6) takes the form:

$$||LF||_{H_K} = ||F||_0. \tag{2.14}$$

Thus $L : H_0 \to H_K$ is an isometry (see (2.14)) and L^* is the left inverse of L (see (2.13)).

We assume that L is injective, that is, the null-space of the operator L is trivial: $N(L) = \{0\}$. Since, by definition, H_K consists of the elements of $R(L)$, that is, $R(L) = H_K$, and L^* is injective on $R(L)$ by (2.13), it follows that

$$L^* = L^{-1}, \tag{2.15}$$

where $L^{-1} : H_K \to H_0$ is a bounded linear operator. The claim is proved.

Formula (2.15) is equivalent to the inversion formula (31) in [Sa2, p.56], while (2.14) is equivalent to formula (33) in [Sa2, p.57].

It is now clear that the assumptions in [Sa1,2] are equivalent to the assumption that $L : H_0 \to H_K$ *is a unitary operator, so that its inverse is* L^*.

This assumption makes the description of the range of L and the inversion formula trivial.

It is suggested in [Sa1] and in [Sa2] to use the norm $||f||_{H_K} = ||L^{-1}f||_0 = ||F||_0$ on $R(L)$, where L is an injective linear operator, and it was claimed in these works that one gets in such a way a characterization of the range of the operator L defined by formula (2.1). In fact this suggestion does not give a nontrivial and practically useful characterization of the range $R(L)$ of this linear integral operator because the norm $||L^{-1}f||_0$ cannot, in general, be described in terms of the usual norms, such as Sobolev or Hoelder norms, for example. Likewise, the fact that the inverse of a unitary operator L is L^* does not give a nontrivial inversion formula, since the main difficulty is to characterize the space H_K in terms of the usual norms (such as Sobolev norms, for example) and to check that $L : H_0 \to H_K$ is a unitary operator.

Finally, one can easily check that if the assumption in [Sa1, p.7] and [Sa2, p.56] holds (this assumption says that H_K has the inner product of $L^2(E, d\mu)$):

$$\int_E \int_E A(p,q) f(p) \overline{g(q)} dp\, dq = \int_E f(p) \overline{g(p)} w(p) dp, \qquad \forall f, g \in H_K, \tag{2.16}$$

where $A(p,q)$ is a nonnegative-definite kernel of the operator K^{-1} (see formula (1.1)), and $w(p)$ is a continuous weight function, $0 < c_0 \leq \nu(p) \leq c_1$, $p \in E$, then

$$A(p,q) = w(p)\delta(p-q),$$

which is an equation similar to (2.12), with $w(p) = v^{-1}(p)$.

This means that the assumption in [Sa2, p.56] that the RKHS H_K is realizable as $L^2(E, d\mu)$ is equivalent to the assumption that the reproducing kernel $K(p,q)$ is of the form (2.12) provided that $d\mu = w(p)dp$.

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